Chance-Constrained Model Predictive Control

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This work focuses on robustness of model-predictive control with respect to satisfaction of process output constraints. A method of improving such robustness is presented. The method relies on formulating output constraints as chance constraints using the uncertainty description of the process model. The resulting on-line optimization problem is convex. The proposed approach is illustrated through a simulation case study on a high-purity distillation column. Suggestions for further improvements are made.

Introduction

Robustness is a highly desirable property for process control systems. Qualitatively speaking, a controller is robust if it results in actual closed-loop behavior that does not deviate unacceptably from the nominal closed-loop behavior, which, in turn, corresponds to a nominal process behavior. For example, a model-based controller results in robust closed-loop stability if the closed loop is stable, even if there is a discrepancy between the model used by the controller and the actual process behavior. The extent of such discrepancy for which closed-loop stability is maintained corresponds to the degree of robustness of that controller. Although necessary, robust stability is usually not sufficient for good controller performance. Other closed-loop properties may have to be maintained in the presence of a discrepancy between the nominal behavior of a process and its actual one. For instance, the resulting regulation error magnitude (such as its 2-norm or ∞-norm) in a feedback loop has to remain "small" in the presence of nominal/actual process behavior discrepancy. Such a requirement is frequently referred to as robust performance. Along with robust stability, robust performance, as defined in the previous sentence, has been studied extensively. However, as just explained, there are many more properties that capture closed-loop performance. One such property, particularly important for constrained model-predictive control (MPC) systems, is the satisfaction of various inequality constraints.

Inequality constrained MPC systems rely on the on-line optimization of an objective function over a future moving horizon, subject to various constraints. At each time step, process measurements are used to formulate the on-line optimization problem whose solution determines an optimal input, which is fed to the process.

The robustness of unconstrained MPC has been studied extensively. Since an unconstrained MPC system is equivalent to a linear time-invariant system, robust unconstrained MPC analysis and synthesis methods relying on either time-domain or frequency-domain descriptions can be used. Discussions of frequency-domain and time-domain methods can be found in Morari and Zafiriou (1989) and Mosca (1995), respectively. For constrained MPC systems, the study of robustness has several facets, and is at a less mature stage. Robust stability results for constrained MPC, within the framework set by Rawlings and Muske (1993), have been presented by a number of investigators, including Genceli and Nikolaou (1993), Michalska and Mayne (1993), Zheng and Morari (1993), Chen and Allgöwer (1998), Lee and Yu (1997), Badgwell (1997), de Nicolao et al. (1998). The purpose of this work is to examine a different aspect of constrained MPC robustness, namely, robustness with respect to satisfaction by the actual system of inequality constraints posed in the on-line optimization problem. While inequality constraints that place bounds on process inputs can be easily satisfied by the actual system, constraints on process outputs are more elusive. That is because future process outputs within an MPC moving horizon have to be predicted on the basis of a process model (involving the effects of manipulated inputs and disturbances on process outputs). Because the model involves uncertainty, process output predictions are also uncertain. This uncertainty in process output predictions may result in adverse violation of output constraints by the actual closed-loop system, even though predicted outputs over the moving horizon might

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have been properly constrained. Consequently, a method of incorporating model uncertainty into the output constraints of the on-line optimization is needed. This would improve the robustness of constrained MPC. In this article, we introduce an approach toward achieving that goal.

The proposed approach relies on formulating output constraints of the type $y_{\min} \leq y \leq y_{\max}$ as chance constraints of the type

$$Pr\{ y_{\min} \leq y \leq {}_{\max}\} \geq \alpha,$$

where $Pr\{A\}$ is the probability of event A occurring, y is the process output bounded by $y_{\min}$ and $y_{\max}$, and α is the specified probability, or confidence level, that the output constraint would be satisfied. Under the assumption that the process output y is predicted by a linear model with normally distributed coefficients, the preceding chance constraint can be reformulated as a convex, deterministic constraint on process inputs. This new constraint can then be readily incorporated into the standard MPC formulation. The resulting on-line optimization problem can be solved using reliable convex optimization algorithms.

The rest of the article is structured as follows. We first provide a brief overview of stochastic programming and chance-constrained optimization. Next, we show how the MPC on-line optimization problem can be cast as a chance-constrained problem. Subsequently, we present an example of using chance-constrained MPC on a high-purity distillation column, an ill-conditioned system. Finally, we draw conclusions and make suggestions for further research.

Stochastic Programming and Chance-Constrained Optimization

Stochastic programming is an optimization technique in which the constraints or objective function of an optimization problem contain stochastic parameters. Chance-constrained optimization is one method of stochastic programming that attempts to reconcile optimization over uncertain constraints (Charnes and Cooper, 1963). The constraints, which contain stochastic parameters, are guaranteed to be satisfied with a certain probability at the optimum found. A typical chance-constrained stochasticprogramming problem has the following form (Birge and Louveaux, 1997):

$$\begin{aligned} &\min_x f(x) \\ \text{s.t. } & g_1(x) \leq 0 \\ & Pr\{ g_2(p,x) \leq 0\} \geq \alpha \end{aligned} \tag{2}$$

where $x \in \Re^n$ is the decision variable vector; $f(x) \in \Re$; $g_1(x) \in \Re^{m_1}$; and $g_2(p,x) \in \Re^{m_2}$ contains the stochastic parameter vector $p \in \Re^p$. If the probability density function of p is known, then the probabilistic constraint $Pr\{g_2(p,x) \leq 0\} \geq \alpha$ can, in principle, be substituted by a deterministic constraint of the form $g_3(x) \leq 0$, so that the entire optimization problem can be handled as an ordinary nonlinear-programming problem.

Depending on the form of g_2, the explicit form of g_3 can be difficult to obtain. The task of developing an explicit closed form for g_3 is greatly simplified if g_2 is affine in the parameter vector p, that is, $g_2(p,x) = A(x)p + b(x)$, where $A(x) \in \Re^{m_2 \times p}$.

For situations in which the stochastic parameters p can be separated from the decision variable x in a constraint, such as

$$Pr\{ \tilde{g}_2(x) \leq p\} \geq \alpha, \tag{3}$$

the deterministic equivalent can be found fairly easily by using the probability density function (pdf) of p to find a value, β, from

$$\alpha = \int \cdots \int_{\beta}^{\infty} \text{pdf}(p)\ dp \tag{4}$$

such that if $\tilde{g}_2(x) \leq \beta$, then Eq. 3 holds.

Linearly constrained problems with normally distributed stochastic parameters can also be handled efficiently. Consider, for example, a constraint of the form

$$Pr\{ a^T(Mx+c) \leq b\} \geq \alpha, \tag{5}$$

where a is a normally distributed vector with mean $\bar{a}$ and covariance P_a. Since $a^T(Mx+c)$ is a linear transformation of a, then it is normally distributed. The mean of $a^T(Mx+c)$ is $\bar{a}^T(Mx+c)$, and its variance

$$\text{Var}\left[a^T(Mx+c)\right] = (Mx+c)^T P_a (Mx+c). \tag{6}$$

Using Eq. 6, we can then rearrange the constraint in Eq. 5 to the standard normal form

$$Pr\left\{ \frac{a^T(Mx+c) - \bar{a}^T(Mx+c)}{\sqrt{(Mx+c)^T P_a (Mx+c)}} \leq \frac{b - \bar{a}^T(Mx+c)}{\sqrt{(Mx+c)^T P_a (Mx+c)}} \right\} \geq \alpha. \tag{7}$$

Then using the value of the confidence level, α, we get

$$\frac{b - \bar{a}^T(Mx+c)}{\sqrt{(Mx+c)^T P_a (Mx+c)}} \geq K_\alpha, \tag{8}$$

where K_α is the value of the inverse cumulative distribution function of the standard normal distribution evaluated at α, usually denoted as $F^{-1}(\alpha)$. Hence the stochastic constraint of Eq. 5 can be recast as the deterministic constraint of Eq. 8. Note that Eq. 8, rewritten as

$$\bar{a}^T(Mx+c) \leq b - K_\alpha \sqrt{(Mx+c)^T P_a (Mx+c)}, \tag{9}$$

indicates that merely replacing a by $\bar{a}$ in Eq. 5 and then replacing Eq. 5 by the deterministic inequality

$$\bar{a}^T(Mx+c) \leq b \tag{10}$$

is not correct, because it can lead to violation of the constraint

$$a_r^T(Mx + c) \leq b, \tag{11}$$

where a_r is a realization of the random variable a, with high probability. As will become clear in the sequel, this observation is important for MPC systems employing output inequality constraints and uncertain models in which parameters appear linearly (such as linear, Volterra, Hammerstein, and Wiener).

Chance-Constrained MPC

Consider a process whose output y must stay below an upper bound $y_{\max}$. This requirement would normally be translated into a set of MPC constraints of the form

$$y(k+i|k) \leq y_{\max}, \qquad i = 1, \ldots, n_c \tag{12}$$

that would have to be incorporated in an optimization problem solved at time k. Process output constraints such as in Eq. 12, included in the MPC on-line optimization problem over a finite horizon, involve prediction of future values of process outputs $y(k+i|k)$. This prediction is made with the use of a model and is never exact. One way to describe the uncertainty in future output predictions is to consider (1) uncertainty in the model describing the effect of manipulated variables on process outputs, and (2) uncertainty in future disturbances. Both kinds of uncertainty are difficult to capture. For example, model uncertainty can be described as parametric uncertainty or structural uncertainty. Similarly, disturbance uncertainty can be described in terms of a stochastic model, but that model might vary drastically with time. Therefore, quantification of the uncertainty of future output predictions cannot possibly capture all possible cases. In this article we focus on a particular case described in detail below.

The future output prediction $y_l(k+i|k)$, made at time k, is given by the linear model

$$y_l(k+i|k) = \sum_{j=1}^{N} h_{l1,j} u_1(k+i-j|k) + \cdots + \sum_{j=1}^{N} h_{lm,j} u_m(k+i-j|k) + d_l(k+i|k). \tag{13}$$

The preceding model can be used for stable processes, for which it can capture dynamics of any order. The prediction made by this model has two sources of uncertainty: (1) the uncertain coefficients $h_{lk,j}$, and (2) the uncertain future disturbance $d_l(k+i|k)$. To simplify the presentation, we will assume that $d_l(k+i|k) = d_l(k|k)$. Moreover, we will assume that $d_l(k|k)$ can be estimated as

$$d_l(k+i|k) = d_l(k|k) = y_l(k) - \sum_{j=1}^{N} h_{l1,j} u_1(k-j) - \cdots - \sum_{j=1}^{N} h_{lm,j} u_m(k-j). \tag{14}$$

In practice, Eq. 14 implies that the disturbance does not contain any high-frequency components, or that these frequencies are filtered out in the feedback path. After substitution and rearrangement using the relationships $s_{lk,i} = \sum_{j=1}^{i} h_{lk,j}$, $i = 1, 2, \ldots,$ between the step- and impulse-response coefficients $s_{lk,j}$ and $h_{lk,j}$, respectively, and $\Delta u_i(k) \triangleq u_i(k) - u_i(k-1)$, the impacts of future control moves and past control moves are separated, as

$$\begin{aligned} y_l(k+i|i) = &\sum_{j=1}^{i} s_{l1,j} \Delta u_1(k+i-j|k) + \cdots + \sum_{j=1}^{i} s_{lm,j} \Delta u_m(k+i-j|k) \\ &+ \sum_{j=1}^{N-1} (s_{l1,i+j} - s_{l1,j}) \Delta u_1(k-j) + \cdots + \sum_{j=1}^{N-1} (s_{lm,i+j} - s_{lm,j}) \Delta u_m(k-j) + y_l(k), \end{aligned} \tag{15}$$

or in vector/matrix notation,

$$\begin{aligned} y_l(k+i|k) = &\left[s_{l1}^T \cdots s_{lm}^T \right] \begin{bmatrix} D_i L & \mathbf{0} & \ddots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \ddots & \mathbf{0} & D_i L \end{bmatrix} \begin{bmatrix} \Delta u_{1,f} \\ \vdots \\ \Delta u_{m,f} \end{bmatrix} \\ &+ \left[s_{l1}^T \cdots s_{lm}^T \right] \begin{bmatrix} F_{i+1} - F_1 & \mathbf{0} & \ddots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \ddots & \mathbf{0} & F_{i+1} - F_1 \end{bmatrix} \begin{bmatrix} \Delta u_{1,p} \\ \ddots \\ \Delta u_{m,p} \end{bmatrix} \\ &+ y_l(k), \end{aligned} \tag{16}$$

where $s_{lj} \in \Re^N$; $\Delta u_{j,f} \in \Re^M$ and $\Delta u_{j,p} \in \Re^{N-1}$ are the future and past input moves, respectively; $p > N$; and the matrices D_i, F_i, and L are as follows:

$$D_i = \begin{bmatrix} I_{i \times i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{N-i \times N-i} \end{bmatrix} \in \Re^{N \times N}$$

$$F_i = \left[\begin{array}{cccc:cccc} \multicolumn{4}{c:}{\mathbf{0}_{i-1 \times p-i}} & \multicolumn{4}{c}{\mathbf{0}_{i-1 \times N-1-p+i}} \\ \hdashline 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 \end{array} \right] \in \Re^{p \times N-1}$$

$$L = \left[\begin{array}{cccc} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \\ \hdashline \multicolumn{4}{c}{\mathbf{0}_{p-M \times M}} \end{array} \right] \in \Re^{p \times M}.$$

If the coefficient vectors s_{lj} are random variables, then by Eq. 16 the output prediction $y_l(k+i|k)$ is also random. Consequently, instead of the constraint in Eq. 12, one has to consider a constraint of the form

$$Pr\{ y_l(k+i|k) \leq y_{\max}\} \geq \alpha, \qquad i=1,\ldots,n_c. \quad (17)$$

Using Eq. 16 to substitute $y_l(k+i|k)$ into the preceding equation yields

$$Pr\left\{s^T\left(A_i \Delta u_f + B_i \Delta u_p\right) \leq y_{l,\max} - y_l(k)\right\} \geq \alpha, \qquad i=1,\ldots,n_c. \quad (18)$$

where

$$s \triangleq \left[s_{l1}^T \cdots s_{lm}^T\right]^T \quad (19)$$

$$A_i \triangleq \begin{bmatrix} D_i L & \mathbf{0} & \ddots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \ddots & \mathbf{0} & D_i L \end{bmatrix} \quad (20)$$

$$B_i \triangleq \begin{bmatrix} F_{i+1} - F_1 & \mathbf{0} & \ddots \\ \mathbf{0} & \ddots & \mathbf{0} \\ \ddots & \mathbf{0} & F_{i+1} - F_1 \end{bmatrix} \quad (21)$$

$$\Delta u \triangleq \begin{bmatrix} \Delta u_{1,f} \\ \vdots \\ \Delta u_{m,f} \end{bmatrix} \quad (22)$$

$$\Delta u_p \triangleq \begin{bmatrix} \Delta u_{1,p} \\ \vdots \\ \Delta u_{m,p} \end{bmatrix}. \quad (23)$$

The preceding equation is of the form of Eq. 5. Assuming that s is normally distributed with mean $\bar{s}$ and covariance P_s, one can use Eq. 9 to convert Eq. 18 to its deterministic counterpart

$$\bar{s}^T\left(A_i \Delta u_f + B_i \Delta u_p\right) \leq y_{l,\max} - y_l(k) - K_\alpha \sqrt{\left(A_i \Delta u_f + B_i \Delta u_p\right)^T P_s \left(A_i \Delta u_f + B_i \Delta u_p\right)}, \qquad i=1,\ \ldots, n_c, \quad (24)$$

where the decision variable is the vector Δu_f.

Remarks

• For the case of a minimum bound on the output, the development of the corresponding chance constraint follows the same path.

• The constraint in Eq. 24 is convex. Indeed, the preceding constraint can be written as

$$\bar{s}^T A_i \Delta u_f + \bar{s}^T B_i \Delta u_p + K_\alpha \left\| P_s^{0.5}\left(A_i \Delta u_f + B_i \Delta u_p\right)\right\|_2 \leq y_{l,\max} - y_l(k), \quad (25)$$

which is the sum of a norm plus a linear function, hence convex. In addition, the previous constraint is deterministic and can be easily incorporated into the standard model-predictive control algorithm.

• When the MPC on-line optimization problem becomes infeasible due to excessively tight output constraints, those constraints can be softened through the introduction of additional softening variables (Zafiriou and Chiou, 1993) as

$$Pr\{ y_l(k+i|k) \leq y_{l,\max} + \epsilon_{l,i}\} \geq \alpha, \qquad i=1,\ldots,n_c. \quad (26)$$

In that case Eq. 18 becomes

$$Pr\left\{ s^T\left(\left[A_i \quad -1\right]\left[\Delta u_f \quad \epsilon_i\right]^T + B_i \Delta u_p\right) \leq y_{l,\max} - y_l(k)\right\} \geq \alpha, \qquad i=1,\ldots,n_c, \quad (27)$$

which, in turn, can be converted to a corresponding deterministic inequality using Eq. 9.

• For an output-constrained MPC system employing a process model with uncertain parameters, a simple deterministic constraint of the form

$$\bar{s}^T\left(A_i \Delta u_f + B_i \Delta u_p\right) \leq y_{l,\max} - y_l(k), \qquad i=1,\ldots,n_c \quad (28)$$

is usually formulated. That constraint fails to account for the uncertainty in the process-model parameters, captured by the last term in Eq. 24. That term involves the covariance matrix P_s of the model parameters, which is usually obtained in standard least-squares identification experiments, along with the parameter estimates. If the matrices $\bar{s}^T(A_i + B_i)$ are not well conditioned, then the effects of inputs Δu_f and Δu_p on outputs will be highly sensitive to uncertainties in the coefficients of these matrices. As we are going to demonstrate in the case study that follows, this can happen in systems with outputs that are highly correlated.

• One alternative for the enforcement of output constraints in the presence of process-model uncertainty would be to simply tighten those bounds, independently of process inputs. However, there are two difficulties with that approach: (1) process output bounds may have to be made excessively tight, thus making the overall closed-loop system unnecessarily conservative, and (2) to avoid that conservatism, output bounds may not be made tight enough, thus resulting in possible violation of output constraints.

Case Study

Process

A continuous-time, 5-state dynamic model of a high-purity distillation process from Skogestad and Postlethwaite (1996), sampled at a rate of 2 min, was used to analyze the chance-constraint formulation. The composition in the two output streams was to be controlled by reflux ratio and boilup rate in an L/V feedback control configuration, as shown in Figure 1. The process model is nearly singular, resulting in poorly conditioned output constraints. The Hessian of the quadratic MPC on-line objective is also poorly conditioned, resulting in a challenging optimization problem.

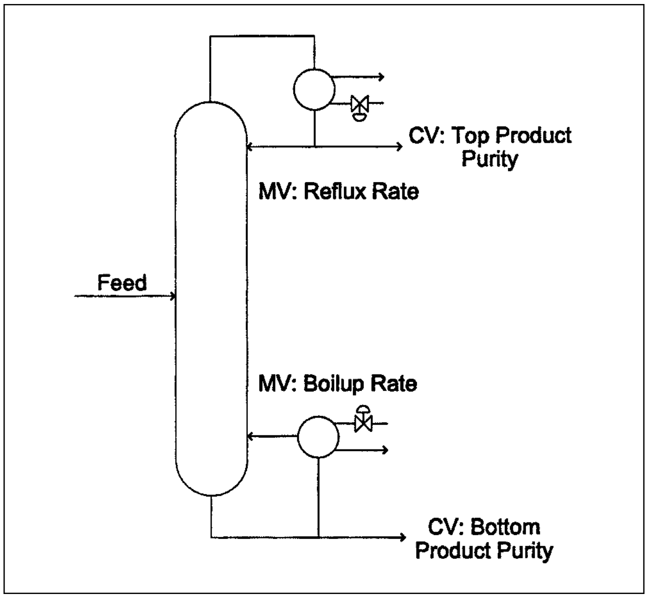

Figure 1. Distillation column.

Identification

The system used in the simulation is a multivariable system with two inputs and two outputs, with n_c chance constraints applied to each individual output. Thus there is a constraint-softening term for each output at each of the n_c predictions into the future, yielding a total of $2n_c$ constraint-softening terms. Each output is predicted on the basis of the two finite impulse-response (FIR) models corresponding to the two inputs. Using a vector representation, the model is arranged as follows:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} s_{11}^T & s_{12}^T \\ s_{21}^T & s_{22}^T \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{29}$$

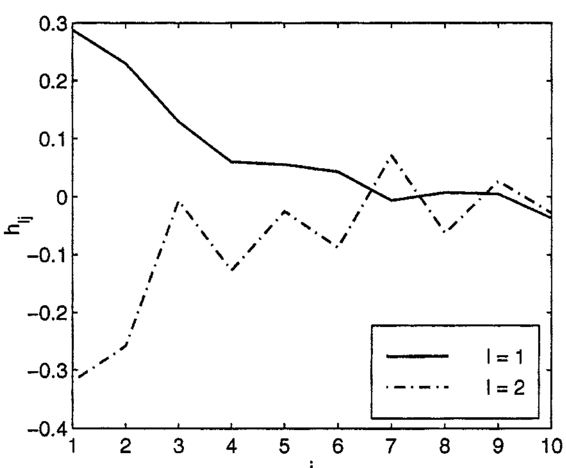

Figure 2a. Impulse response coefficients for relating the inputs to output 1 (200 I/O, SNR = 15).

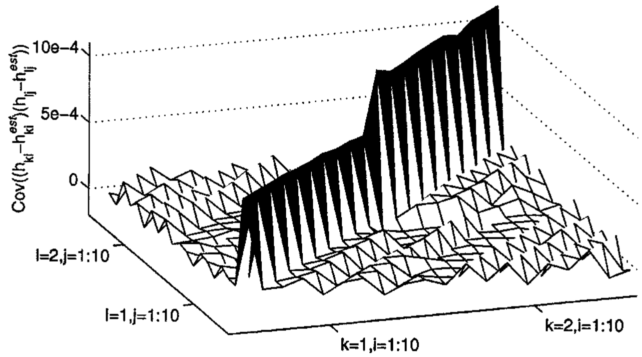

Figure 2b. Covariance matrix for impulse response coefficient estimates of Figure 2a.

To develop a process model and uncertainty description for the purpose of demonstrating our method, we generated output data from the aforementioned state-space model using a pseudorandom binary sequence (PRBS) input with an ampli-

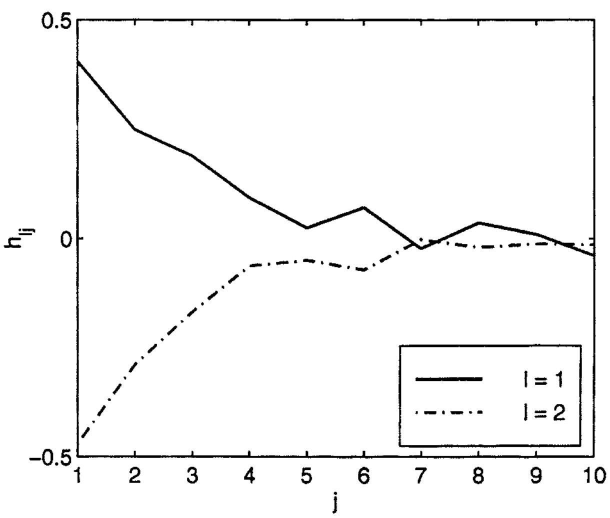

Figure 3a. Impulse response coefficients for relating the inputs to output 2 (200 I/O, SNR = 18).

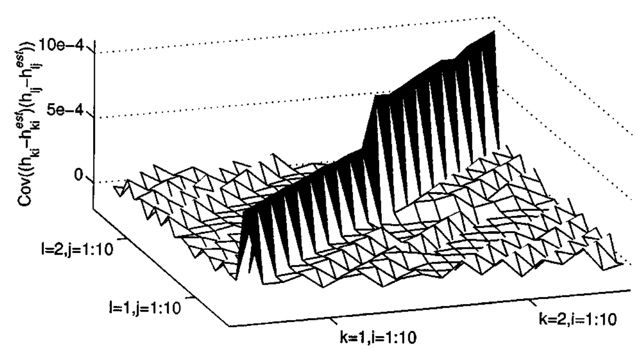

Figure 3b. Covariance matrix for impulse response coefficient estimates of Figure 3a.

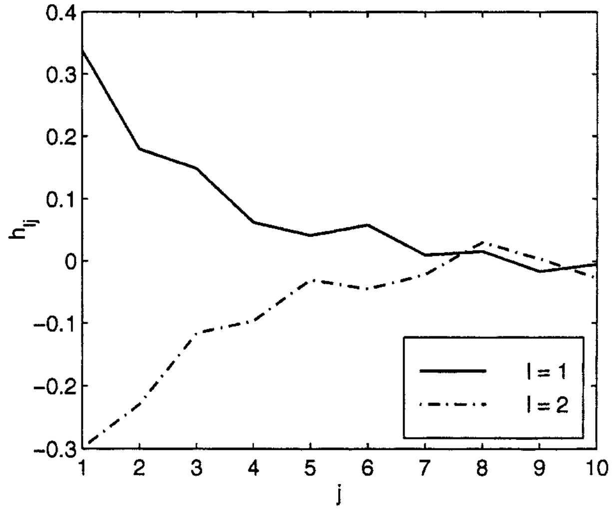

Figure 4a. Impulse response coefficients for relating the inputs to output 1 (500 I/O, SNR = 13).

tude of 1. The output data were then corrupted with normally distributed noise with a given signal-to-noise ratio (SNR) between 11 and 18. The SNR was defined as

$$\mathrm{SNR} = 10\log_{10}[\mathrm{variance(output)}/\mathrm{variance(noise)}]. \quad (30)$$

Standard least-squares techniques were then used to identify multi-input−single-output (MISO) FIR models for each output using the corrupted data. Because the objective of this case study was to demonstrate how chance constraints can handle a given level of modeling uncertainty, no further sophistication was used in estimating model parameters.

Three models were generated that show the benefits of chance-constrained model-predictive control. Each model was identified using a different number of input−output data points—200, 500 and 1,000—and similar SNRs, yielding models with differing levels of uncertainty. Figures 2 through 7 show the impulse-response coefficients and their corresponding covariance matrices. Each of these models was then utilized in the creation of two model predictive controllers, the first corresponding to standard output-constrained MPC, and the second corresponding to MPC with chance constraints.

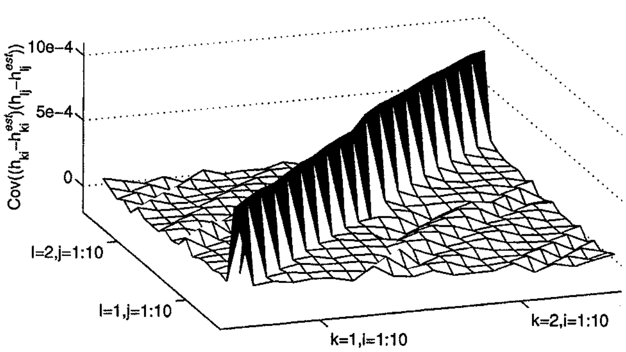

Figure 4b. Covariance matrix for impulse response coefficient estimates of Figure 4a.

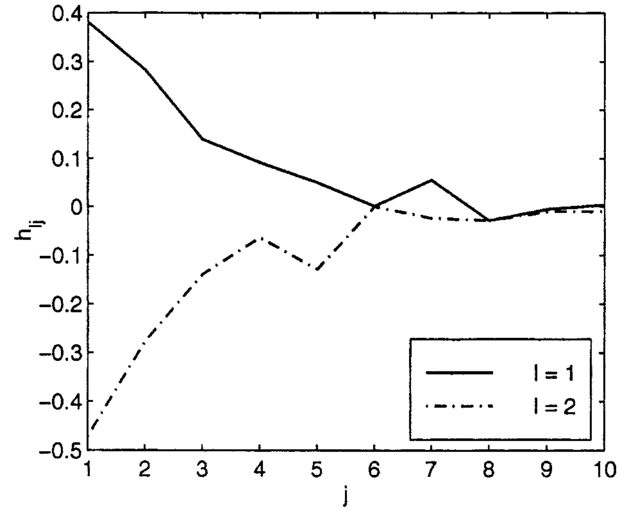

Figure 5a. Impulse response coefficients for relating the inputs to output 2 (500 I/O, SNR = 16).

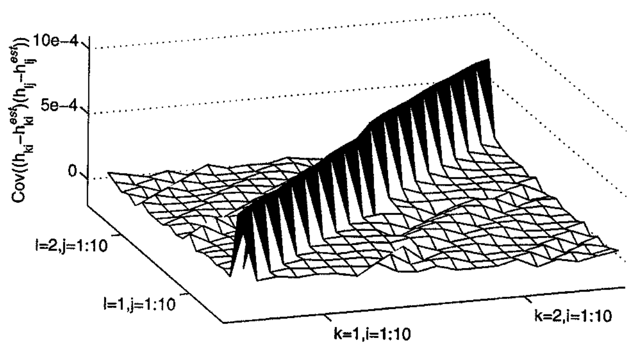

Figure 5b. Covariance matrix for impulse response coefficient estimates of Figure 5a.

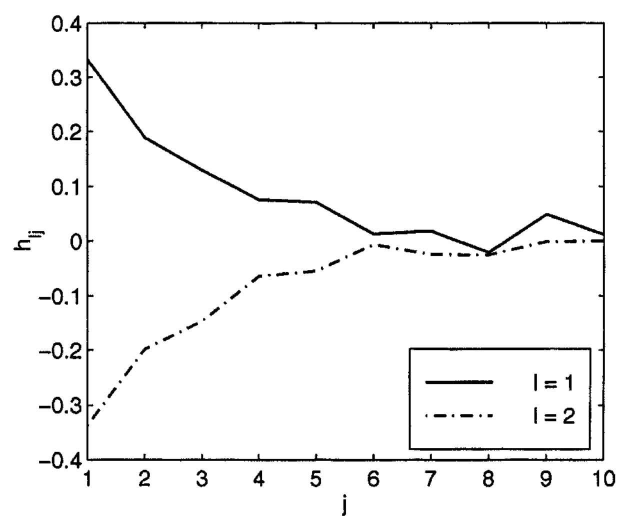

Figure 6a. Impulse response coefficients for relating the inputs to output 1 (1,000 I/O, SNR = 11).

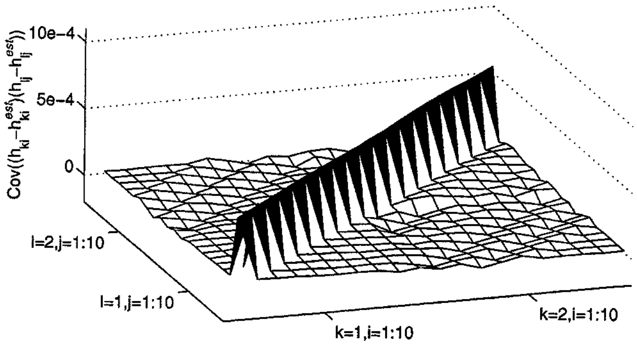

Figure 6b. Covariance matrix for impulse response coefficient estimates of Figure 6a.

Chance constraints

The input directions of maximum uncertainty for the three models can be determined from the following optimization problem, according to Eq. 24:

$$\max_{x} \quad x^T P x$$
$$\text{s.t.} \quad -\mathbf{1} \le x(i) \le \mathbf{1}, \qquad i = 1,2,$$

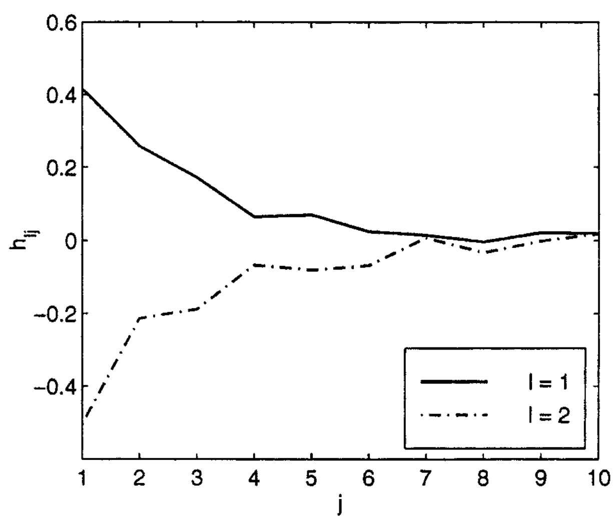

Figure 7a. Impulse response coefficients for relating the inputs to output 2 (1,000 I/O, SNR = 13).

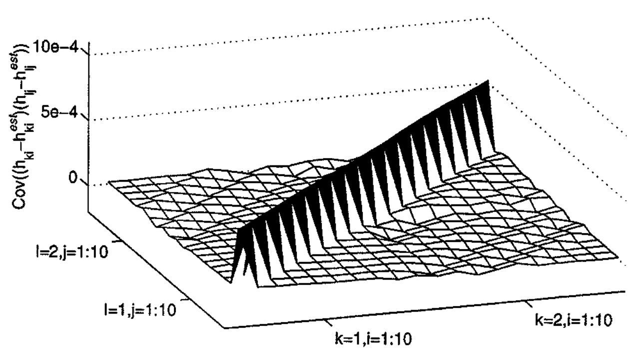

Figure 7b. Covariance matrix for impulse response coefficient estimates of Figure 7a.

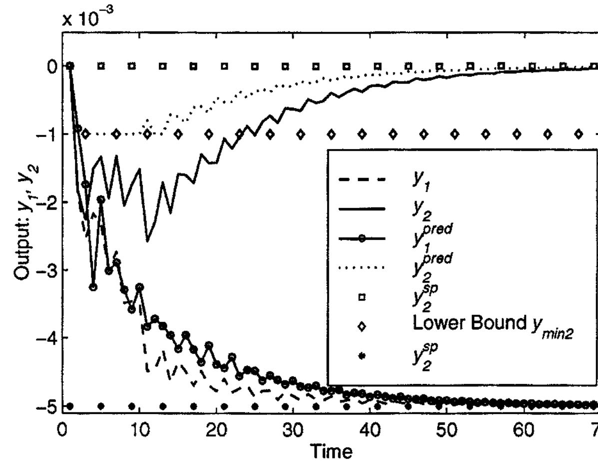

Figure 8. Process response to setpoint change $y^{sp} = -5e-3$, standard MPC used with model developed from 200 I/O points.

where $x = [u_1(k)\ u_2(k)]^T$ and P is the Hessian for each of the models identified:

$$P_{200} = \begin{bmatrix} 7.65 \times 10^{-4} & -2.95 \times 10^{-5} \\ -2.95 \times 10^{-5} & 1.14 \times 10^{-3} \end{bmatrix}$$

$$P_{500} = \begin{bmatrix} 6.70 \times 10^{-4} & 4.83 \times 10^{-5} \\ 4.83 \times 10^{-5} & 7.93 \times 10^{-4} \end{bmatrix}$$

$$P_{1,000} = \begin{bmatrix} 5.28 \times 10^{-4} & 2.44 \times 10^{-6} \\ 2.44 \times 10^{-6} & 5.41 \times 10^{-4} \end{bmatrix}.$$

The problem to be solved is the maximization of a convex function over a convex set (equivalent to the minimization of a concave function over a convex set) that, in general, is *NP-hard* (Horst and Tuy, 1990). The solutions of such problems

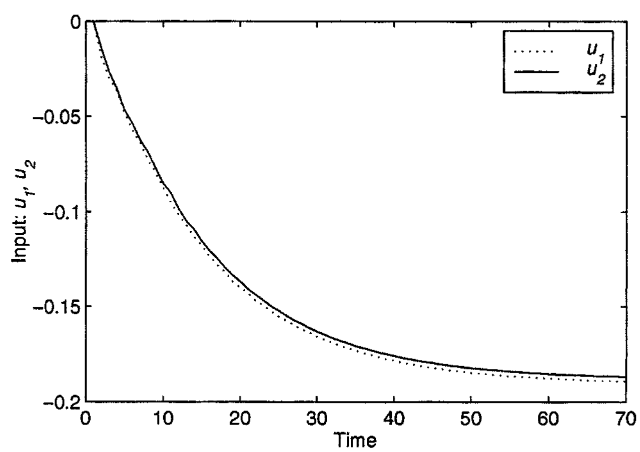

Figure 9. Process input to setpoint change $y^{sp} = -5e-3$, standard MPC used with model developed from 200 I/O points.

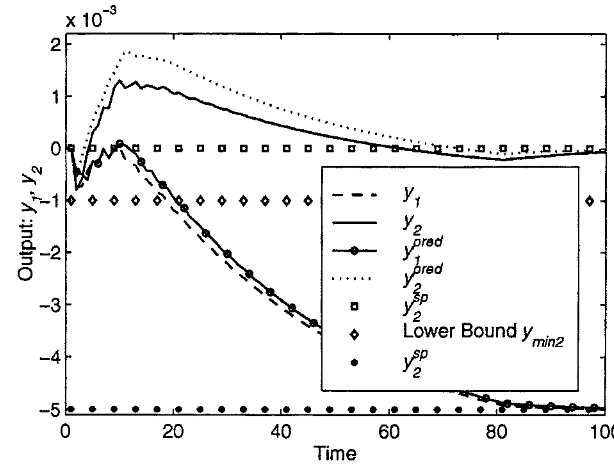

Figure 10. Process response to setpoint change $y^{sp} = -5e-3$, chance-constrained MPC used with model developed from 200 I/O points.

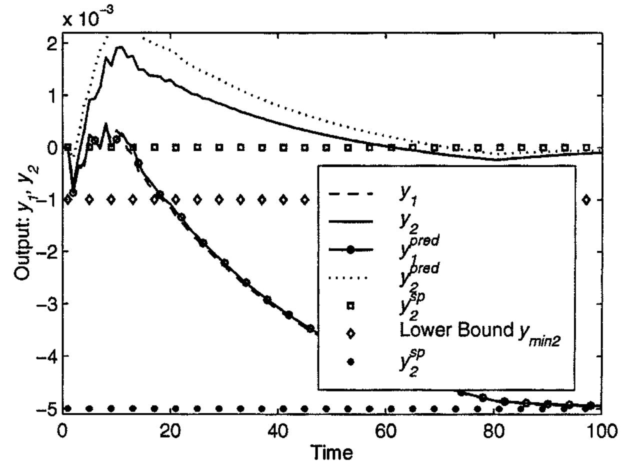

Figure 12. Process response to setpoint change $y^{sp} = -5e-3$, chance-constrained MPC used with model developed from 500 I/O points.

are always found on the boundary of the feasible region. For a problem with very small dimensionality and symmetry, this problem does not present any significant difficulty. The problem can be solved using standard constrained nonlinear optimization techniques, and the symmetry of the constraints and objective can be used to determine any degenerate solutions to the problem. The optimization gives solutions at the four corners of the feasible region: $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$; depending on the starting point of the optimization problem. By inspecting the values of the objective function at these points, the true optima can be determined. The first objective, $x^T P_{200} x$, predicts the optima to be at the points $(1,-1)$ and $(-1,1)$. The two subsequent objectives show the optima to be at the points $(1,1)$ and $(-1,-1)$. Due to the gains and the ill-conditioning of the system, the optima for the latter two objectives agree with the nature of the system. This can be see in Figures 2 to 7. Since the cross-correlation for the variables is rather small, the smallest data set actually estimated the cross-correlation to be negative, thus the change in direction is due to the stochastic nature of the identification.

Results

Results for standard MPC are presented in Figures 8 and 9, while results for chance-constrained MPC using the three models are shown in Figures 10 through 15. These figures show the closed-loop responses of the two outputs (top and bottom compositions) and inputs (reflux rate and boilup rate) to a setpoint step change in the top temperature, output 1, of magnitude -5×10^{-3}. The bottom temperature, output 2, was constrained by a lower bound of -1×10^{-3}, and for the chance-constrained MPC the probability of constraint violation was set to 0.01. Figure 8 shows that application of standard MPC to the system resulted in frequent and large violations of the output bound. In contrast, application of

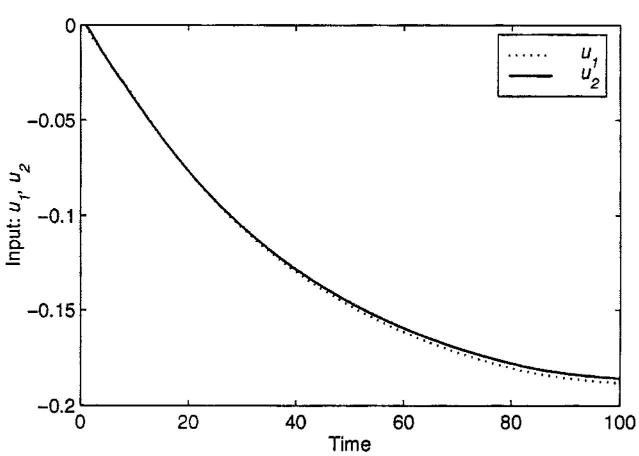

Figure 11. Process input to setpoint change $y^{sp} = -5e-3$, chance-constrained MPC used with model developed from 200 I/O points.

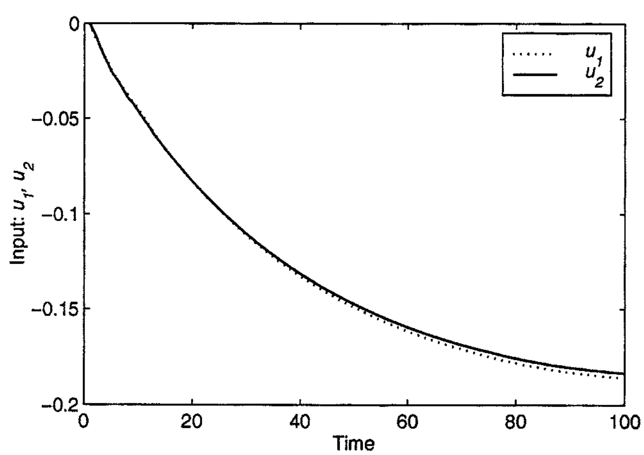

Figure 13. Process input to setpoint change $y^{sp} = -5e-3$, chance-constrained MPC used with model developed from 500 I/O points.

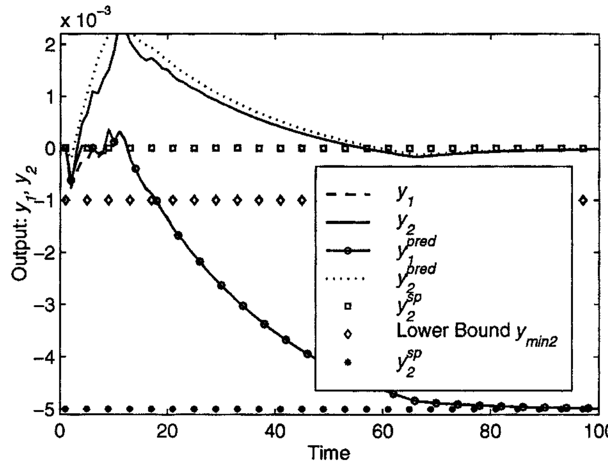

Figure 14. Process response to setpoint change $y^{sp} = -5e-3$, chance-constrained MPC used with model developed from 1,000 I/O points.

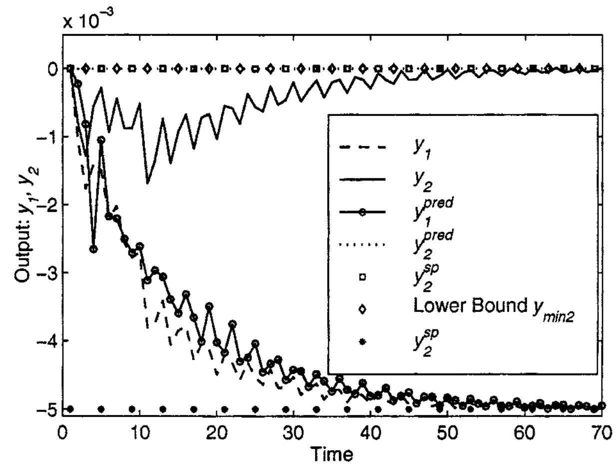

Figure 16. Process input to setpoint change $y^{sp} = -5e-3$, standard MPC with tightened constraints used with model developed from 200 I/O points.

chance-constrained MPC succeeded in preventing violation of the output bound. All simulations were accomplished in the MATLAB environment using the Numerical Analysis Group (NAG) nonlinear-constrained optimizer and the controller described in Table 1.

One way of attempting to deal with the problems associated with standard output-constrained MPC is to tighten the bounds placed on the process outputs. Figures 16 and 17 show how this procedure can still lead to output bounds violation. Process outputs are predicted based on the process model. Since this model is uncertain, the output can be predicted to be within bounds, while in actuality the output violates the bounds. In the case examined here, the bound is close to the setpoint, so that even placing the bound at the setpoint causes

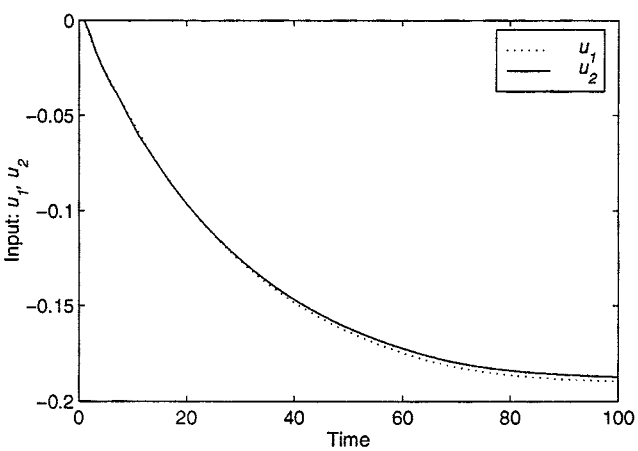

Figure 15. Process input to setpoint change $y^{sp} = -5e-3$, chance-constrained MPC used with model developed from 1,000 I/O points.

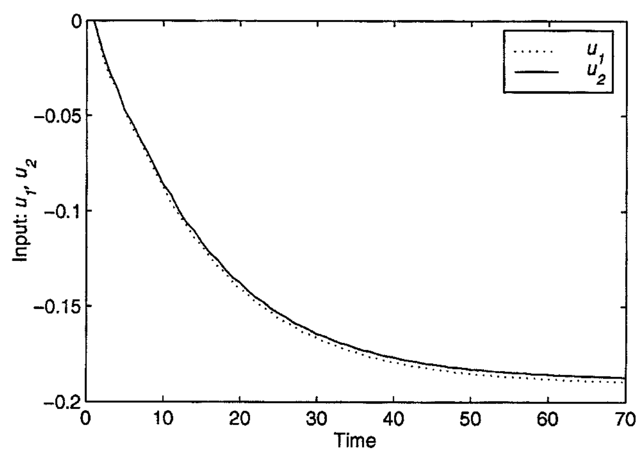

Figure 17. Process input to setpoint change $y^{sp} = -5e-3$, standard MPC with tightened constraints used with model developed from 200 I/O points.

Table 1. MPC and Chance-Constrained MPC Parameter Values

MPC Terms	$r\sum_{i=0}^{M-1}\Delta u(k+i\mid k)^2$		$\sum_{i=1}^{p}[y(k+i\mid k)-y^{SP}]^2$		$\Pr\left\{\begin{matrix} y_{\min}-\epsilon \le y \\ y \le y_{\max}+\epsilon \end{matrix}\right\} \ge \alpha$			$q\epsilon^2$
MPC Parameter	M	r	p	y^{SP}	$y_{\min}$	$y_{\max}$	α	q
Standard output-constrained MPC	4	0.02	14	-5×10^{-3}	-10^{-3}	∞	N/a	10^3
Chance-constrained MPC	4	0.02	14	-5×10^{-3}	-10^{-3}	∞	99%	3

bounds violation. Note that placing the bound above the setpoint would cause offset.

Note that in all simulation output constraints were softened only when infeasibilities would result without softening.

Discussion and Future Research

In this work, we focused on robustness of constrained MPC with respect to satisfaction of process output constraints by a closed-loop MPC system that employs an uncertain process model. We proposed to enhance MPC robustness by formulating process output constraints as chance constraints. Simulations comparing the performance of the chance-constrained MPC formulation vs. that of standard MPC with output constraints showed the ability of the proposed approach to improve the robustness of MPC with respect to output constraint satisfaction. For the high-purity distillation column, a nearly singular system, studied in the simulations, the original output constraints were violated frequently and to a large degree when standard MPC was used, even with extremely large penalties on the constraint-softening variable ϵ. With the use of chance constraints, violation of output constraints was drastically reduced or eliminated.

While chance output constraints were used in this work within a standard MPC framework, there are several possibilities for using chance constraints in other control frameworks, such as stabilization of uncertain processes or combined MPC and identification, where process modeling uncertainty can be naturally incorporated in the on-line optimization problem. In addition, uncertainty in future output predictions due to stochastic disturbances may also be incorporated in the proposed framework.

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